Problem solving session – QISS

Fabio Costa

18/7/2019

The goal of this session is to prove that the memory in an open system with Ising-like interactions can always be simulated classically. This will allow us to practice the calculation of process matrices for a physically motivated scenario.

The model is a chain of n two-level systems (qubits) that interact according to the Hamiltonian

$$H(J) = -J \sum_{j=1}^{n-1} Z^j Z^{j+1}, \tag{1}$$

where the operator Z^j acts as the Pauli matrix $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on the j-th qubit and as the identity on the other qubits, while J is a coupling constant with the dimension of energy. We treat the first qubit as the system and the rest as the environment. We consider the general case of N time steps, namely of N possible sequential operations on the first qubit, separated by arbitrary times $t_1, t_2, \ldots, t_N$. In other words, the first operation takes place at times t_1, the second at time $t_1 + t_2$, and so on. (Note, the number N of time steps is unrelated to the number n of qubits.)

The task is to construct the general process matrix $W_{J,\psi}(t_1, \ldots, t_N)$ and show that it can be decomposed as a sum of positive product processes. To make sure we know what we are looking for, recall that for each time step j we have a pair of Hilbert-Schmidt spaces, $A^j_{IO} \equiv A^j_I \otimes A^j_O$, corresponding to input and output of the local operation. The output of the last time step can be taken to be trivial, because the causal order conditions imply that the process matrix has to be identity there (the output of the last operation cannot influence any other time step). Therefore, our process matrix lives in the space $A^1_{IO} \ldots A^{N-1}_{IO} A^N_I$ (where we omit the tensor product symbol). Each A^j_I and A^j_O is a single-qubit space, so the process matrix corresponds to a $2N-1$-qubit state, which is a $2^{2N-1} \times 2^{2N-1}$ matrix.

The simplifying feature of the Hamiltonian (1) is that it is diagonal in the computational basis. To make things easier to write, we label $|1\rangle$, $|-1\rangle$ the basis states for a single qubit, so that $\sigma_z|\mu\rangle = \mu|\mu\rangle$, $\mu = \pm 1$. The eigenbasis of H is then given by the product states $|\vec{\mu}\rangle \equiv |\mu_1, \ldots, \mu_n\rangle$:

$$H(J)|\vec{\mu}\rangle = -J \left(\sum_{j=1}^{n-1} \mu_j \mu_{j+1} \right) |\vec{\mu}\rangle. \tag{2}$$

Note that the coupling J simply sets a scale for the time evolution. Indeed, the unitary evolution for a time t is $e^{-iH(J)t} = e^{iJt \sum_{j=1}^{n-1} Z^j Z^{j+1}} =$

$e^{i\tilde{t}\sum_{j=1}^{n-1} Z^j Z^{j+1}}$, with $\tilde{t} \equiv Jt$. This extends to the whole multi-time process: $W_{J,\psi}(t_1, \ldots, t_N) = W_{1,\psi}(\tilde{t}_1, \ldots, \tilde{t}_N)$. Therefore, we set $J = 1$ (and stop writing it) and work with dimensionless time steps, which we write again as t_j, dropping the tilde for simplicity.

We can solve the problem in steps. The task is to write the object at each step in the energy eigenbasis.

1. Write the time evolution operator $U(t) = e^{-iHt}$. This is a unitary operator acting on n qubits. It can help to write explicitly the systems on which it acts: $U^{A^1E^2\ldots E^n}(t)$, where A^1 denotes the system qubit and $E^2, \ldots, E^n$ denote the environment qubits.

2. Compose two such evolutions for the environment qubits $2, \ldots n$. This should give us a unitary operator

$$U^{A^1A^2E^2\ldots E^n}(t_1, t_2) = U^{A^2E^2\ldots E^n}(t_2)U^{A^1E^2\ldots E^n}(t_1)$$

that acts on $n+1$ qubits. More precisely, the inputs to this unitary are the n initial qubits (system and environment), plus the output of the operation performed at time t_1. The outputs are all the qubits after time t_2 and the input to the system operation at time t_1.

3. Iterate the step above to write the unitary

$$U^{A^1A^2\ldots A^NE^2\ldots E^n}(t_1, \ldots t_N) = U^{A^NE^2\ldots E^n}(t_N)\cdots U^{A^1E^2\ldots E^n}(t_1)$$

for N time steps.

4. Apply the above unitary to the initial system-environment state $|\psi\rangle^{A^1E^2\ldots E^n} = \sum_{\vec{\mu}} \psi_{\vec{\mu}} |\vec{\mu}\rangle^{A^1E^2\ldots E^n}$, to obtain the isometry

$$V_\psi^{A^1A^2\ldots A^NE^2\ldots E^n}(t_1, \ldots t_N) \\ = U^{A^1A^2\ldots A^NE^2\ldots E^n}(t_1, \ldots t_N)|\psi\rangle^{A^1E^2\ldots E^n}.$$

Note that the notation here is slightly imprecise, as V and U have the same labels, although they have different dimension. Indeed, U has an input and an output space for each of its labels, while V has inputs only for $A^2, \ldots A^N$, while for $A^1, E^2, \ldots E^N$ it only has outputs. Physically, this isometry takes as input the outputs of each local operation on the system, while its outputs are the inputs to all local operations, plus the final environment qubits.

5. Write the Choi representation of the isometry above, $[[V]] \equiv |V\rangle\!\rangle\langle\!\langle V|$. (Of course, it is convenient to use the eigenbasis of the Hamiltonian to define the Choi-Jamiołkowski isomorphism.) Making the distinction between inputs and outputs explicit, the matrix $[[V]]$ lives in the space $A^1_{IO}\ldots A^{N-1}_{IO}A^N_I E^2_I \ldots E^n_I$.

6. The process matrix for the system alone is obtained by tracing out the environment:

$$W_\psi^{A^1_{IO}\ldots A^{N-1}_{IO}A^N_I}(t_1, \ldots t_N) = \mathrm{tr}_{E^2_I\ldots E^n_I}\left[[V_\psi(t_1, \ldots t_N)]\right]^{A^1_{IO}\ldots A^{N-1}_{IO}A^N_I E^2_I\ldots E^n_I}.$$

7. Finally, we have to show that $W_{\psi}^{A_{IO}^{1}\ldots A_{IO}^{N-1}A_{I}^{N}}(t_1,\ldots t_N)$ is a sum of product matrices for any initial state $|\psi\rangle$ and for arbitrary time steps $t_1,\ldots t_N$.